MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 16: Generating Functions II: Products and Catalan Numbers

In this lecture we continue with ordinary generating functions. We will see that if we have certain combinatorial interpretation for the coefficients of two generating functions, then there is a natural way to interpret the coefficients of the product of such functions. We will also see some applications of this interpretation.

Proposition 1. *Let $F(x)$ and $G(x)$ be the generating functions of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{\geq 0}$, respectively. Then the following statements hold.*

1. $F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.
2. $F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$.
3. $F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n$, *where* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Proof. (1) Integrating the formal power series $\sum_{n=0}^{\infty} a_n x^n$ term by term, we obtain

$$F'(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} a_n n x^{n-1}.$$

(2) Adding the for$F(x) + G(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$.

(3) Observe that the coefficient of x^n in the product $F(x)G(x)$ are the sum of the terms $(a_k x^k)(b_\ell x^\ell)$ such that $k + \ell = n$. Therefore

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

□

Example 2. Let C_n be the number of strings of balanced parentheses of length $2n$, where $C_0 = 1$. Let us find a explicit formula for C_n by using generating functions. First, observe that every string of balanced parentheses of length $2n$ has the form $(P_k)P_{(n-1)-k}$ for some $k \in [\![0, n-1]\!]$, where P_k is a string of balanced parentheses of length $2k$, the closing parenthesis ")", which is in $2(k+1)$-th position, is the match of the first opening parenthesis, and $P_{(n-1)-k}$ is a string of balanced parenthesis of length $2((n-1)-k)$. Therefore

$$C_n = C_0 C_{n-1} + \cdots + C_{n-1} C_0 = \sum_{k=0}^{n-1} C_k C_{(n-1)-k}. \tag{0.1}$$

Now let $C(x)$ be the generating function of the sequence $(C_n)_{n \ge 0}$, that is, $C(x) = \sum_{n=0}^{\infty} C_n x^n$. Using the recurrence (0.1), we see that

$$C(x) - 1 = \sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n-1} C_k C_{(n-1)-k} \Big) x^n = x \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} C_k C_{n-k} \Big) x^n = x C(x)^2.$$

Then we see that $xC(x)^2 - C(x) + 1 = 0$, and solving this quadratic equation for $C(x)$ we obtain the solutions $\frac{1}{2x}\big(1 \pm \sqrt{1-4x}\big)$. The equality $C(0) = C_0 = 1$ allows to conclude that

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

By virtue of the Generalized Binomial Theorem,

$$(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = \sum_{n=0}^{\infty} \frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2} \cdots \frac{-(2n-3)}{2}}{n!} x^n = 1 - 2x - \sum_{n=2}^{\infty} \frac{2^n (2n-3)!!}{n!} x^n,$$

where $(2n+1)!! := 1 \cdot 3 \cdots (2n+1)$ for every $n \in \mathbb{N}_0$. Thus,

$$\frac{1 - \sqrt{1-4x}}{2x} = 1 + \sum_{n=2}^{\infty} \frac{2^{n-1}(2n-3)!!}{n!} x^{n-1} = 1 + \sum_{n=1}^{\infty} \frac{2^n (2n-1)!!}{(n+1)!} x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Hence we conclude that $C_n = \frac{1}{n+1}\binom{2n}{n}$ for every $n \in \mathbb{N}$.

Practice Exercises

Exercise 1. *If $F(x)$, $G(x)$, and $H(x)$ are the generating functions of the sequences $(a_n)_{n \ge 0}$, $(b_n)_{\ge 0}$, and $(c_n)_{\ge 0}$, respectively. Argue that*

$$F(x)G(x)H(x) = \sum_{n=0}^{\infty} \Big(\sum_{j+k+\ell=n} a_j b_k c_\ell \Big) x^n.$$

Generalize the previous statement for the product of m generating functions $F_1(x), \dots, F_m(x)$.

Exercise 2. *Let a_n be the number of ways to provide change for n cents out of pennies, nickels, dimes, and quarters.*

(1) *Find the generating function of the sequence $(a_n)_{n \ge 0}$.*
(2) *Find a_{211}.*

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